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INCIDENCE SPACES AND ALGEBRAS

by

E. C. Dade and K. Goldberg

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**California Institute of Technology

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Incidence Spaces and Algebras

E. C. Dade and K. Goldberg*

An incidence matrix is a matrix whose entries are either 0 or 1. We use a fixed notation for the incidence matrices I, the identity matrix, and J, the matrix with 1 in every position. The fact that I, J-I are disjoint incidence matrices which form a basis for a linear algebra, lies behind much of the manipulation of the celebrated "incidence equation"

$$AA^T = kI + \lambda(J-I)$$

in which A is an incidence matrix of order v. For given integers v,k,λ the existence of A is equivalent to the existence of a v,k,λ design, special cases of which are finite projective planes and Hadamard matrices.

Starting in 1955 several mathematicians, including R. C. Bose, D. Mesner and the authors, began considering more general algebras with a basis of incidence matrices whose sum was J. At first each of these matrices was symmetric. Bose, and later Mesner, considered the properties of a design with incidence matrix A such that

$$AA^T = k_1A_1 + k_2A_2 + \dots + k_dA_d$$

$$J = A_1 + A_2 + \dots + A_d$$

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where the A_i are incidence matrices which form the basis of an algebra. We came upon a similar generalization through the graph problem mentioned at the end of this note. In each case $A_1 = I$.

Our purpose is to summarize the results we obtained in generalizing these concepts by not demanding $A_i^T = A_i$ or $A_1 = I$, or even that the basis be an algebra basis.

1. Incidence Spaces. A set $\mathcal{B} = \{A_1, A_2, \ldots, A_d\}$ of m x n incidence matrices is called an <u>incidence basis</u> if

$$A_1 + A_2 + \cdots + A_d = J_{m,n},$$

where $J_{m,n}$ is the m x n matrix with 1 in every position. This implies that the A_p are mutually disjoint (i.e., if A_p has 1 in the i,j position then $A_q \neq A_p$ has 0 in the i,j position), and that for each i and j there is a unique p such that the i,j element of A_p is 1. Thus the A_p are linearly independent.

The linear span

$$\mathfrak{A} = [A_1, A_2, \ldots, A_d]$$

of the incidence basis $\mathcal{B}$ (over some unspecified field $\mathcal{K}$ of characteristic 0) is called an m x n <u>incidence space</u>. The only incidence matrices in $\mathfrak{A}$ are the 2^d sums of distinct elements of $\mathcal{B}$. Therefore, $\mathcal{B}$ is the only incidence basis which is a vector basis for $\mathfrak{A}$, so that we may speak of <u>the</u> incidence basis of an incidence space.

If m = n, so that the A_p are square, and if $\mathfrak{A}$ is closed under matrix multiplication, it is called an <u>incidence algebra</u>. In this case there exist non-negative integers $a_{pq}^{(r)}$, p, q, r = 1, 2...,d such that

$$A_p A_q = \sum_{r=1}^{d} a_{pq}^{(r)} A_r \qquad p,q=1,2,\ldots,d.$$

These integers are called <u>the</u> structure constants of $\mathfrak{A}$.

Let $\mathfrak{A} = [A_1, A_2, \ldots, A_d]$ be an $m \times n$ incidence space closed under right multiplication by $J_{n,n}$. Then for each $p = 1,2,\ldots,d$ the non-zero rows of A_p have the same row sums a_p, and hence the same number of non-zero elements. Let U_p be the set of indices of the non-zero rows of A_p. Then U_p and U_q are either disjoint or equal. We let u_p denote the order of U_p. If $\mathfrak{A}$ is closed under left multiplication by $J_{m,m}$ we similarly define the column sets V_p of order v_p.

An $n \times n$ incidence algebra is closed under both right and left multiplication by $J = J_{n,n}$, hence it has both row and column sets. Any basis element is concentrated in one row set and one column set, and the number of non-zero elements in any two rows (columns) are equal. For a fixed U_p and V_q let M_{pq} denote the sum of those A_r with $U_r = U_p$ and $V_r = V_q$. Then the set of M_{pq} is an incidence basis whose linear span is a direct summand of $\mathfrak{A}$. $\mathfrak{A}$ is simple if and only if it equals this summand and has a unit element.

An incidence space is called stochastic if each basis element has constant row sums and constant column sums. In that case it has just one row set and just one column set.

An $n \times n$ incidence space is called semi-stochastic if each row set U_p is also a column set V_p, (then each column set is a row set). In this case let $\mathcal{C}_{pq}$ be the subspace spanned by those A_r with row set U_p and column set U_q. Then $\mathcal{C}_{pq}$ is isomorphic to a

stochastic $u_p \times u_q$ incidence space, called an associated incidence space, and

$$\mathcal{C}_{pq} \cdot \mathcal{C}_{rs} \subseteq \begin{cases} \mathcal{C}_{ps} & \text{if } q = r \\ \{0\} & \text{if } q \neq r \end{cases}$$

for all p, q, r, s. Conversely, given any set of stochastic $u_p \times u_q$ incidence spaces $\mathcal{C}_{pq}$ satisfying the above multiplicative condition, we can construct a semi-stochastic incidence space having these as its associated incidence spaces.

If an incidence space contains the identity matrix I and is closed under left and right multiplication by J, then it is semi-stochastic.

2. Semi-Symmetric Incidence Algebras. The incidence algebra $\mathfrak{A} = [A_1, A_2, \ldots, A_d]$ is called semi-symmetric if it is closed under matrix transposition; i.e., if for each $p = 1, 2, \ldots, d$ there is a p' such that $A_p^T = A_{p'}$. A semi-symmetric incidence algebra is clearly semi-stochastic and, algebraically, semi-simple.

After a suitable simultaneous renumbering of rows and columns, which does not alter the incidence or algebraic properties of the algebra, certain of the basis elements have the form

$$A_1 = \begin{pmatrix} \overbrace{J_{a_1} \; J_{a_1} \cdots J_{a_1}}^{n_1} & \\ & 0 \cdots 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} \overbrace{0 \cdots 0}^{n_1} & & \\ & \overbrace{J_{a_2} \cdots J_{a_2}}^{n_2} & \\ & & 0 \cdots 0 \end{pmatrix} \qquad \cdots$$

$$A_s = \begin{pmatrix} 0_0 \cdots 0 & \\ & \overbrace{J_{a_s} \cdots J_{a_s}}^{n_s} \end{pmatrix}$$

where the empty spaces are zeros. The row and column sets of $\mathfrak{A}$ are the row sets $U_1, U_2, \ldots, U_s$ of $A_1, A_2, \ldots, A_s$ respectively.

If A_p has row set U_i and column set U_j, then A_p has various submatrices equal to J_{a_i, a_j} arranged as entries in an $n_i \times n_j$ incidence matrix; e.g.,

$$A_p = \begin{array}{c} \overbrace{\hphantom{J'\;J'\;J'}}^{U_j} \\ \left(\begin{array}{c|ccc|c} & & & & \\ \hline & J' & J' & 0 & \\ & 0 & J' & J' & \\ & J' & 0 & J' & \\ \hline & & U_j & & \end{array}\right)\left.\vphantom{\begin{array}{c}J'\\0\\J'\end{array}}\right\} U_i \end{array} \qquad J' = J_{a_i,a_j}$$

again with zeros in the blanks. If we replace the J' by 1 (and the $0_{a_i,a_j}$ by 0) we arrive at a new $(n_1+n_2+\ldots+n_\delta) \times (n_1+n_2+\ldots+n_\delta)$ incidence algebra, also semi-symmetric, which contains the identity I. These two algebras are clearly isomorphic. Thus the study of semi-symmetric incidence algebras may be reduced to the study of those containing I.

The unit element of $\mathfrak{A}$ is clearly $a_1^{-1}A_1 + a_2^{-1}A_2 + \ldots + a_\delta^{-1}A_\delta$. One may also prove that if an arbitrary incidence algebra $\mathfrak{A} = [A_1, A_2, \ldots, A_d]$ contains a unit element of the form $x_1A_1 + x_2A_2 + \ldots + x_dA_d$ with each x_i non-negative, then $\mathfrak{A}$ is equivalent, in the above sense, to an incidence algebra containing I.

An incidence algebra is called <u>symmetric</u> if each of the basis matrices are symmetric, and <u>anti-symmetric</u> if semi-symmetric but the only symmetric basis matrices are those with ones on the main diagonal. A symmetric incidence algebra is commutative and stochastic.

A stochastic, semi-symmetric incidence algebra is symmetric if and only if all the basis matrices have only real characteristic roots.

In an arbitrary incidence algebra the subset of those matrices whose transposes are also in the algebra is a semi-symmetric incidence algebra. The subset of symmetric matrices is an incidence space, and an algebra if and only if it is commutative.

3. Characters of Commutative Incidence Algebras. In this section we assume that $\mathcal{K}$ is algebraically closed. Let $\mathcal{O}\!\mathcal{L} = [A_1, A_2, \ldots, A_d]$ be a commutative $n \times n$ incidence algebra. It is stochastic because each A_p commutes with $J = J_{n,n}$.

For any $\lambda_1, \ldots, \lambda_d$ in $\mathcal{K}$ let $\mathcal{M}(\lambda_1, \ldots, \lambda_d)$ denote the maximal space of $n \times 1$ vectors over $\mathcal{K}$ such that for each $p=1,2,\ldots,d$

$$(A_p - \lambda_p I)^{\mu_p} \cdot \mathcal{M}(\lambda_1, \ldots, \lambda_d) = \{0\}$$

with μ_p some positive integer. Since $\mathcal{O}\!\mathcal{L}$ is commutative, $\mathcal{O}\!\mathcal{L} \cdot \mathcal{M}(\lambda_1, \ldots, \lambda_d)$ is contained in $\mathcal{M}(\lambda_1, \ldots, \lambda_d)$.

There are only a finite number of $\mathcal{M}(\lambda_1, \ldots, \lambda_d)$ different from $\{0\}$. Index those with not all $\lambda_i = 0$ by $\mathcal{M}_1, \ldots, \mathcal{M}_t$, with

$$\mathcal{M}_k = \mathcal{M}(\lambda_1^{(k)}, \ldots, \lambda_d^{(k)}) \qquad k=1,2,\ldots,t$$

and let $\mathcal{M}_0 = \mathcal{M}(0,\ldots,0)$. Note that $\mathcal{M}_0 = \{0\}$ if and only if $\mathcal{O}\!\mathcal{L}$ contains I. The set $\mathcal{M}$ of all $n \times 1$ vectors over $\mathcal{K}$ is the direct sum of $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_t$. The maximal subspace of $\mathcal{M}$ annihilated by a power of $A_p - \lambda I$ is the direct sum of those $\mathcal{M}_k$ with $\lambda_p^{(k)} = \lambda$.

The non-zero characters of $\mathcal{O}\!\mathcal{L}$ are the t homomorphisms of $\mathcal{O}\!\mathcal{L}$ onto $\mathcal{K}$:

$$\chi_k : \sum_{p=1}^{d} x_p A_p \to \sum_{p=1}^{d} x_p \lambda_p^{(k)} \qquad k=1,2,\ldots,t.$$

If we denote the dimension of $\mathfrak{M}_k$ by v_k, then v_k is called the <u>multiplicity</u> of χ_k. We have $v_o + v_1 + \ldots + v_t = n$, with v_o non-negative and the other v_k positive integers. If σ is any automorphism of $\mathcal{K}$, then χ_k^σ maps A_p into $(\chi_k(A_p))^\sigma$ and thus is a character of $\mathfrak{A}$. Denote it by $\chi_{\sigma(k)}$. Then $v_{\sigma(k)} = v_k$. Thus, if $\lambda_p^{(i)}$ and $\lambda_p^{(j)}$ are roots of the same irreducible polynomial over a ground field of $\mathcal{K}$ then $v_i = v_j$.

Since $\mathfrak{A}$ is stochastic, for each $p = 1, 2, \ldots, d, a_p$ is a characteristic root of A_p (the one of largest absolute value) and $e = (1\ 1\ \ldots\ 1)^T$ is a characteristic vector. Since the characteristic root $n = a_1 + \ldots + a_d$ of J is not multiple, one of the $\mathfrak{M}_k$, say $\mathfrak{M}_1$, is just the scalar multiples of e. That is, $v_1 = 1$ and $\chi_1(A_p) = \lambda_p^{(1)} = a_p$ for all p.

Let

$$\Lambda = (\lambda_i^{(j)})$$

denote the d x t matrix of characteristic roots of the A_p as defined above. Since J has one characteristic root equal to n and the others equal to 0, the column sums of Λ are, in order, n, 0,..., 0.

Let

$$D_v = \mathrm{diag}(v_1, \ldots, v_t).$$

Then the trace of A_p is the i-th row sum of ΛD_v. Let t_{pq} denote the trace of $A_p A_q$, and let

$$T = (t_{ij})$$

Then

$$T = \Lambda D_v \Lambda^T. \tag{3.1}$$

Let $e_1, \ldots, e_t$ denote the primitive idempotents of $\mathfrak{A}$. Then $\chi_i(e_j) = \delta_{ij}$. Let μ_{ip} be defined by $e_i = \sum_{p=1}^{d} \mu_{ip} A_p$, and let M denote the t x d matrix

$$M = (\mu_{ij}).$$

Then

$$M\Lambda = I_t, \tag{3.2}$$

the identity matrix of order t. Using this in (3.1) we get

$$TM^T D_v^{-1} = \Lambda \tag{3.3}$$

From these equations we have

$$\text{rank } T = \text{rank } \Lambda = t$$

$$\text{nullity } T = d-t = \text{dimension of radical of } \mathfrak{A}$$

Thus T and Λ are non-singular if and only if $d = t$ or, equivalently, $\mathfrak{A}$ is semi-simple. In that case (3.1) yields

$$(\det \Lambda)^2 = (\det T) / \prod_{i=1}^{t} v_i \tag{3.4}$$

Suppose $\mathfrak{A}$ is semi-symmetric, and let $A_p^T = A_{p'}$ as before. Then $\chi_i(A_{p'}) = \overline{\chi_i}(A_p)$ or

$$\lambda_{p'}^{(i)} = \bar{\lambda}_p^{(i)}$$

where the bar denotes complex conjugation. In this case $\mathfrak{A}$ is semi-simple and (3.4) becomes

$$(\det \Lambda)^2 = (-1)^{s/2} n^d \prod_{p=1}^{d} a_p / \prod_{i=1}^{d} v_i$$

where s is the number of $p \neq p'$. In this case $\prod v_i$ must divide $n^d \prod a_p$. We also have $\mu_{ip} = v_i \lambda_{p'}^{(i)}/na_p$ and the two equalities

$$\sum_{i=1}^{d} v_i \lambda_p^{(i)} \lambda_{q'}^{(i)} = \delta_{pq} na_p \qquad p,q=1,2,\ldots,d$$

$$\sum_{i=1}^{d} v_i \lambda_p^{(i)} \lambda_q^{(i)} \lambda_{r'}^{(i)} = na_r a_{pq}^{(r)} \qquad p,q,r=1,2,\ldots,d.$$

4. Group Generation. Let G be a group of n x n permutation matrices. Let $\mathcal{A}(G)$ be the vector space of all matrices in $\mathcal{K}_n$ commuting with every element of G. Then $\mathcal{A}(G)$ is an incidence algebra, called a group-generated incidence algebra. We let $\mathcal{P}(G)$ denote the algebra spanned by G over $\mathcal{K}$.

Let P be the permutation group on $\{1,2,\ldots,n\}$ corresponding to G. Then (x_{ij}) is an element of $\mathcal{A}(G)$ if and only if

$$x_{\pi(i)\pi(j)} = x_{ij} \qquad i,j=1,2,\ldots,n$$

for all π in P. We shall say that G is transitive if P is transitive, and generally speak of the permutations in G as if they were the corresponding permutations in P. The elements of the incidence basis of $\mathcal{A}(G)$ are the incidence matrices of the equivalence sets of the equivalence: $(i,j) \sim (i',j')$ if and only if there is an element of P taking i into i' and j into j'. That is, the i,j element of the p-th incidence matrix is 1 if and only if (i,j) is in the p-th equivalence set.

A fundamental fact about $\mathcal{A}(G)$ is that it and $\mathcal{P}(G)$ are centralizers of each other and of no larger subsets in $\mathcal{K}_n$. The center of each is their intersection, and is spanned by the matrices formed by summing the elements in each conjugacy class of G.

If $\mathcal{A}$ is an arbitrary incidence algebra we let $G(\mathcal{A})$ denote the set of permutation matrices which commute with every element of $\mathcal{A}$. Then $\mathcal{A}(G(\mathcal{A}))$ contains $\mathcal{A}$, and $\mathcal{A}(G')$ contains $\mathcal{A}$ if and only if $G(\mathcal{A})$ contains G'. Thus $\mathcal{A}$ is group-generated if and only if it is generated by $G(\mathcal{A})$. Also if $G(\mathcal{A})$ is a proper subgroup of G'' then $\mathcal{A}(G'')$ is a proper subalgebra of $\mathcal{A}(G(\mathcal{A}))$.

Some of the properties of $\mathfrak{A}(G) = [A_1, A_2, \ldots, A_d]$ are

1) $\mathfrak{A}(G)$ is semi-symmetric (and thus semi-simple) and contains I.

2) $\mathfrak{A}(G)$ is stochastic if and only if G is transitive.

3) $\mathfrak{A}(G)$ is commutative if and only if it is contained in $\mathcal{P}(G)$.

4) $\mathfrak{A}(G)$ is anti-symmetric if and only if G is of odd order.

5) $\mathfrak{A}(G)$ is symmetric if and only if each transposition occurs as a disjoint cycle in G.

6) The row and column sets of $\mathfrak{A}(G)$ are the transitivity sets of G.

7) If G_i is the subgroup of G leaving i fixed then $a_p = [G_i : G_i \cap G_j]$ for any i, j such that the i, j element of A_p is 1.

8) If G is transitive then d is the number of transitivity sets of $G_i^{(i)}$.

9) The order of the row set U_p is $[G : G_i]$ for any i in U_p.

10) $\mathfrak{A}(G) = \mathcal{P}(G)$ if and only if G is transitive and abelian.

11) If G contains a full-cycle permutation the elements of $\mathfrak{A}(G)$ are polynomials in that matrix.

12) If Q is a permutation matrix $\mathfrak{A}(QGQ^{-1}) = Q\,\mathfrak{A}(G)Q^{-1}$.

13) If $\mathcal{P}(G)$ is written as a sum of irreducible representations Γ_i of dimension k_i: $\mathcal{P}(G) = n_1 \Gamma_1 + n_2 \Gamma_2 + \ldots + n_r \Gamma_r$, then $n = \sum_{i=1}^{r} n_i k_i$, $d = \sum_{i=1}^{r} n_i^2$, and $\mathfrak{A}(G)$ is isomorphic to $\mathcal{H}_{n_1} + \mathcal{H}_{n_2} + \ldots + \mathcal{H}_{n_r}$.

14) $\mathfrak{A}(G) = [I, J-I]$ if and only if G is doubly transitive.

As an illustration of the applications of these properties consider the following. If G is doubly transitive then $\mathfrak{A}(G) = [I, J-I]$. The centralizer of this $\mathfrak{A}(G)$ is the set of matrices which commute with J, the set of all matrices with constant row and column sum. The centralizer of $\mathfrak{A}(G)$ is also $\mathcal{P}(G)$. Therefore any matrix with constant row and column sums is a linear combination of matrices taken from a doubly transitive group of permutation matrices.

We use a similar method to construct an incidence <u>space</u> S(G,H) from a <u>pair</u> of groups of permutation matrices G,H whose corresponding permutation groups P,Q are representations of degrees m and n respectively of some finite group. To each element M of G and π of P we denote by M^{τ} and π^{τ} the corresponding elements of H and Q respectively.

The S(G,H) has a similar set of three equivalent definitions as does $\mathfrak{A}(G)$ above: S(G,H) is the set of all m by n matrices $X = (x_{ij})$ such that (i)

$$x_{\pi(i),j} = x_{i,\pi^{\tau}(j)} \qquad i = 1,2,\ldots,m;\ j = 1,2,\ldots,n$$

for all π in P; (ii) $PX = XP^{\tau}$ for all P in G; and (iii) S(G,H) has as a basis the set of incidence matrices of the equivalence sets of the equivalence: $(i,j) \sim (i',j')$ if and only if there is an element π of P such that π takes i into i' and π^{τ} takes j' into j.

Among the properties of S(G,H) are: it is closed under left multiplication by J_m and under right multiplication by J_n; the

transposes form the incidence space $S(H,G)$; if G is transitive then the dimension of $S(G,H)$ is equal to the number of transitivity sets of G_i^{τ} and the row sums of the incidence basis are the orders of these transitivity sets; and if U,V,W are any elements of $S(G,H)$ then UV^TW is an element of $S(G,H)$. It follows that the set $\{UV^T : U,V \in S(G,H)\}$, which is a subset of $\mathfrak{A}(G)$, is closed under multiplication.

An example of the results of this method is the following. Let K be a group with a subgroup M of double index 2 (i.e. $K = M + M \times M$) and index v. Let N be any other subgroup of K of index v. Then a v,k,λ design exists with k equal to the index of $M \cap N$ in M.

What amounts to this method is described by J. S. Frame in [4]. His primary interest was in the derivable group properties, so that his results and ours do not intersect.

5. Dimension 3 with identity. Suppose the incidence algebra $\mathfrak{A}$ has dimension 3 and contains I. Then $\mathfrak{A}$ is stochastic and commutative, and we may write

$$\mathfrak{A} = [I, A, J-I-A].$$

Let n be the order of A, a its constant row and column sum, and c_1, c_2, c_3 non-negative integers such that

$$A^2 = c_1 I + c_2 A + c_3 (J-I-A).$$

The other structure constants of $\mathfrak{A}$, each of which is a non-negative integer, may be determined from

$$A(J-I-A) = (a-c_1)I + (a-c_2-1)A + (a-c_3)(J-I-A)$$

$$(J-I-A)^2 = (N-2a-1+c_1)I + (n-2a+c_2)A + (n-2a-2+c_3)(J-I-A)$$

By interchanging A and J-I-A, if necessary, we can assume that

$$a \leq \frac{n-1}{2}$$

Then the c_i are non-negative integers satisfying

$$a \geq c_1,\ c_2 + 1,\ c_3 \geq 0$$

and $c_3 \geq 1$ if $a = (n-1)/2$.

A is normal if and only if $\mathfrak{A}$ is either symmetric or anti-symmetric. $\mathfrak{A}$ is symmetric if and only if $c_1 = a$, and anti-symmetric if and only if $c_1 = 0$.

$\mathfrak{A}$ is always semi-simple, and therefore has three characters χ_1, χ_2, χ_3. Set $a = \chi_1(A)$ and let $\lambda_2 = \chi_2(A)$ and $\lambda_3 = \chi_3(A)$. The characteristic roots λ_2, λ_3 have multiplicities v_2, v_3 and satisfy the equation

$$x^2 + (c_3-c_2)x + c_3 - c_1 = 0$$

We also have

$$v_2 + v_3 = n-1$$

$$v_2\lambda_2 + v_3\lambda_3 = -a$$

and

$$(\lambda_2 - \lambda_3)^2 v_2 v_3 = n(c_1(n-1)-a^2)$$

If λ_2, λ_3 are irrational then $a = v_2 = v_3 = (n-1)/2$ and either (1) $\mathfrak{A}$ is symmetric, $n \equiv 1 \pmod 4$, $A^2 + A = ((n-1)/4)(J+I)$, $\lambda_2, \lambda_3 = (-1 \pm n^{\frac{1}{2}})/2$ or (2) $\mathfrak{A}$ is anti-symmetric; $n \equiv 3 \pmod 4$, $A^2 + A = ((n+1)/4)(J-I)$, $\lambda_2, \lambda_3 = (-1 \pm in^{\frac{1}{2}})/2$. This latter is the only way for $\mathfrak{A}$ to be anti-symmetric. It leads to the equation

$$AA^T = \frac{n+1}{4} I + \frac{n-3}{4} J$$

which means that A is a v,k,λ design from which one can construct a Hadamard matrix of order n+1. From our method of the previous section (see properties 4 and 8), a sufficient condition for A to exist is that there be a transitive permutation group on n letters, of odd order, with the subgroups leaving one letter fixed having exactly three transitivity sets. Unfortunately, when n is not a prime power such a group would be primitive, and therefore insoluble of odd order, contrary to the classical conjecture.

Now assume that λ_2 and λ_3 are rational. Then one is negative and the other non-negative. Let

$$s = \lambda_2 \geq 0 \text{ and } t = -\lambda_3 \geq 1.$$

We can show that if $\mathfrak{A}$ is semi-symmetric it is symmetric, that n is never a prime, that tr $A^2 \geq a^2 + at$ with equality if and only if A is singular, that $A^T \neq A$ implies $a^2 + a \geq n$, that $s \neq a$ implies $s \leq a + 2 - 2(a+1)^{\frac{1}{2}}$ and $t \neq a$ implies $t \leq (a+1)/2$.

6. A graph theory problem. Given $2t$ points, consider the set $\mathcal{L}$ of all (linear, undirected) graphs on these points consisting of t disjoint lines. That is, in each graph each point is the neighbor of exactly one other point. There are

(6.1) $$n = (2t-1)(2t-3)\ldots 3\cdot 1$$

such graphs:

$$\mathcal{L} = \left\{ L_1, L_2, \ldots, L_n \right\}.$$

The union of two elements of $\mathcal{L}$ is a collection of disjoint cycles each having an even number of points. A cycle with $2k$ points is said to be of length k. Thus the lengths of the cycles form a partition of t.

Given a partition $\pi = (1^{j_1}, 2^{j_2}, \ldots, t^{j_t})$ of t:

$$t = j_1 + 2j_2 + \ldots + tj_t,$$

let $\mathcal{H}_\pi$ denote the set of all graphs on the $2t$ fixed points have j_k cycles of length k for $k = 1, 2, \ldots, t$. Let R be the set of partitions of t:

$$R = \left\{ \pi_1, \pi_2, \ldots, \pi_d \right\} \qquad d = p(t).$$

where $p(t)$ denotes the number of partitions of t.

Given fixed elements L of $\mathcal{L}$ and $\pi_p = (1^{j_1}, 2^{j_2}, \ldots, t^{j_t})$ of R, it is easy to show that there are

$$a_p = 2^{t-j_1-\cdots-j_t}\frac{t!}{1^{j_1}j_1!\,2^{j_2}j_2!\cdots t^{j_t}j_t!}$$

elements of $\mathcal{L}$ whose union with L lies in $\mathcal{H}_{\pi_p}$.

The following related question is more difficult to answer. Given fixed elements L_i and L_j of $\mathcal{L}$, and π_p and π_q of Π, for how many elements L_k of $\mathcal{L}$ do we have

(6.2) $$L_i \cup L_k \in \mathcal{H}_{\pi_p} \text{ and } L_k \cup L_j \in \mathcal{H}_{\pi_q} ?$$

It can be shown that the answer $a_{pq}^{(r)}$ only depends upon π_p, π_q, and π_r where

(6.3) $$L_i \cup L_j \in \mathcal{H}_{\pi_r}.$$

To solve this problem, we shall construct an incidence algebra whose structure constants are the above mentioned $a_{pq}^{(r)}$.

Let $\mathcal{S}_{2t}$ be the set of permutations of the 2t fixed points. Let H_1 be the subgroup of $\mathcal{S}_{2t}$ leaving L_1 fixed. Let

$$H_1, H_2, \ldots, H_n$$

be the left cosets of H_1 in $\mathcal{S}_{2t}$, such that any element of H_i takes L_1 into L_i. This is the same n as in (6.1).

For any element σ of $\mathcal{S}_{2t}$ we have

$$H_i = H_{\gamma_\sigma(i)} \qquad i = 1, 2, \ldots, n$$

where γ_σ is a permutation of the integers 1, 2,..., n. Let

$$P = \{\gamma_\sigma : \sigma \in \mathcal{S}_{2t}\}$$

be the group of all permutations γ_σ. Let G be the group of permutation matrices of order n corresponding to P.

The incidence algebra we ~~are look for~~ seek is $\mathfrak{A}(G)$.

The proof of this depends on two facts: (F_1) any element of $\mathcal{S}_{2t}$ permutes the elements of $\mathcal{H}_\pi$, for any π in P; and (F_2) if $L_i \cup L_j$ and $L_{i'} \cup L_{j'}$ are both in $\mathcal{H}_\pi$, then there is an element of $\mathcal{S}_{2t}$ taking L_i into $L_{i'}$ and L_j into $L_{j'}$.

Let us restate (F_1) and (F_2) in terms of ordered pairs of the cosets $H_1, H_2, \ldots, H_n$, and the equivalence:

$$(H_i, H_j) \sim (H_{i'}, H_{j'})$$

if and only if there is an element of P taking i into i' and j into j'. Then (F_1) and (F_2) imply that the equivalence sets $\overline{\mathcal{H}}_{\pi_1}, \overline{\mathcal{H}}_{\pi_2}, \ldots, \overline{\mathcal{H}}_{\pi_d}$ of this equivalence correspond to the sets $\mathcal{H}_{\pi_1}, \mathcal{H}_{\pi_2}, \ldots, \mathcal{H}_{\pi_d}$ in such a way that (H_i, H_j) is in $\overline{\mathcal{H}}_{\pi_p}$ if and only if $L_i \cup L_j$ is in $\mathcal{H}_{\pi_p}$.

Let $A_p = (a_{pij})$ be the incidence matrix of $\overline{\mathcal{H}}_{\pi_p}$. That is,

$$a_{pij} = \begin{cases} 1 \text{ if } (H_i, H_j) \in \overline{\mathcal{H}}_{\pi_p} \\ 0 \text{ otherwise} \end{cases}$$

From our discussion in section 4, we know that A_p is one of the elements of the incidence basis of $\mathfrak{A}(G)$. If $\{a_{pq}^{(r)}\}$ is the set of structure constants of this basis we have

$$A_p A_q = \sum_{r=1}^{d} a_{pq}^{(r)} A_r.$$

Comparing the i, j elements we have

$$\sum_{k=1}^{n} a_{pik}\, a_{qkj} = a_{pq}^{(r)}$$

where

$$a_{rij} = 1.$$

Since $a_{pik}\, a_{qkj} = 1$ if and only if $a_{pik} = a_{qkj} = 1$, the constant $a_{pq}^{(r)}$ is just the number of H_k such that

$$(H_i, H_k) \in \overline{\mathcal{H}}_{\pi_p} \text{ and } (H_k, H_j) \in \overline{\mathcal{H}}_{\pi_q}$$

where

$$(H_i, H_j) \in \overline{\mathcal{H}}_{\pi_r}.$$

Comparing these conditions with (6.2) and (6.3), we see that the structure constant $a_{pq}^{(r)}$ is indeed the answer to our problem.

That this is a reasonable solution follows from that fact that we can actually construct the incidence basis of $\mathfrak{A}(G)$ given the group P. In turn this group only depends upon the group H_1 which leaves the graph L_1 fixed.

This same method applies to any set of graphs $\{L_i\}$ which are permuted among themselves by permutations of their points, so long as the sets $\{\mathcal{H}_i\}$ into which the unions of L_i are partitioned satisfy conditions corresponding to (F_1) and (F_2).

Under any conditions the incidence algebra $\mathfrak{A}(G)$ is symmetric (and therefore commutative and stochastic) because $L_i \cup L_j = L_j \cup L_i$, and thus $a_{pij} = a_{pji}$ for all p, i, j. Furthermore, the characteristic roots of its incidence basis matrices are all rational integers. This follows from property 3 in section 4, the fact that P is a representation of a symmetric group, and the fact that the characters of any representation of a symmetric group are rational (see [5], vol. II, pages 190-193).

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